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# Spectral Chebyshev–Fourier collocation for the Helmholtz and variable coefficient equations in a disk

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ABSTRACT

The paper is concerned with the spectral collocation solution of the Helmholtz equation in a disk in the polar coordinates $r$ and $\theta$. We use spectral Chebyshev collocation in $r$, spectral Fourier collocation in $\theta$, and a simple integral condition to specify the value of the approximate solution at the center of the disk. The scheme is solved at a quasi optimal cost using the idea of superposition, a matrix decomposition algorithm, and fast Fourier transforms. Both the Dirichlet and Neumann boundary conditions are considered and extensions to equations with variable coefficients are discussed. Numerical results confirm the spectral convergence of the method.



## 1. Introduction

The numerical solution of the Helmholtz equation in a disk with spectral methods has attracted a lot of attention in the last few decades, see, for example, [6–8,10,11], and references therein. Some of the proposed spectral Galerkin [10,11] and tau [7] methods are quasi optimal; a method is quasi optimal if its cost (the number of required arithmetic operations) is $O(N^2 \log N)$ while the number of unknowns is $O(N^2)$. The purpose of this paper is to show that a particular spectral collocation method, essentially the same method as the one proposed in [8], is also quasi optimal. We accomplish this goal by solving the resulting linear system directly using the idea of superposition, a matrix decomposition algorithm, and fast Fourier transforms (FFTs). (The resulting linear system in [8] is solved using a preconditioned iterative method with a finite difference preconditioner.) In general, the formulation of collocation methods is much simpler than the formulation of Galerkin and tau methods, in particular for equations with variable coefficients. Moreover, collocation methods do not involve evaluations or approximations of integrals. Unlike the methods proposed in [7,11], the collocation method of the present study is not based on an odd–even parity approach. As observed, for example, in [8], the collocation method of [8] is more accurate than the method in [7] and may also give good results for exact solutions which are not infinitely differentiable. In fact, it had been observed even earlier in [6] that in comparison with methods based on the parity approach, improved center resolutions can be achieved by expanding in $2r - 1$ (in the case of the unit disk) rather than in $r$ and by using the complete set of Chebyshev polynomials. We believe that our spectral collocation algorithm is competitive with the spectral Galerkin method of [10]. The method of [10] is also based on the diagonalization approach, performed at the continuous rather than the discrete level, and it involves a more subtle treatment of singularity at $r = 0$ by considering two cases ($m \neq 0$ and $m = 0$) for the solution of the resulting one-dimensional equations in $r$.

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An outline of the paper is as follows: in Section 2, we formulate the Dirichlet problem for the Helmholtz equation in a disk in the polar coordinates $r$ and $\theta$ and re-derive the integral condition which is used later to obtain the value of the approximate solution at the center of the disk. In Section 3, we define our spectral collocation scheme for the Dirichlet problem. The scheme uses Chebyshev collocation in $r$ and Fourier collocation in $\theta$. The idea of superposition and a matrix decomposition algorithm with FFTs for solving the collocation scheme is described in Section 4. In Section 5, we consider the corresponding scheme and algorithm for the Neumann problem. In Section 6, the algorithms are extended to the Dirichlet and Neumann problems for equations with variable coefficients. Numerical results are presented in Section 7 and concluding remarks are given in Section 8.

## 2. Preliminaries

Let $D$ be the disk with radius $R > 0$ centered at the origin, that is, $D = \{(x,y): x^2 + y^2 \leqslant R^2\}$. Consider the Dirichlet problem for the Helmholtz equation in $D$ which consists in finding a twice continuously differentiable function $s$ such that

$$\Delta s - \alpha s = p(x,y), \ (x,y) \in D, \quad s = q(x,y), \ (x,y) \in \partial D, \tag{2.1}$$

where $p$ and $q$ are given functions and $\alpha$ is a given non-negative constant. Let $\Omega$ be the rectangle $(0,R) \times (0,2\pi)$, and let the functions $u$, $f$, and $g$ be defined by

$$u(r,\theta) = s(r\cos\theta, r\sin\theta), \tag{2.2}$$

$$f(r,\theta) = r^2 p(r\cos\theta, r\sin\theta), \quad g(\theta) = q(R\cos\theta, R\sin\theta). \tag{2.3}$$

Then it is easy to verify that

$$Lu = f(r,\theta), \quad (r,\theta) \in \Omega, \tag{2.4}$$

$$u(0,\theta) = \text{const}, \quad u(R,\theta) = g(\theta), \ \theta \in [0,2\pi], \tag{2.5}$$

$$u(r,0) = u(r,2\pi), \quad u_\theta(r,0) = u_\theta(r,2\pi), \ r \in (0,R), \tag{2.6}$$

$$\int_0^{2\pi} u_r(0,\theta)\mathrm{d}\theta = 0, \tag{2.7}$$

where

$$Lu = r^2 u_{rr} + r u_r + u_{\theta\theta} - \alpha r^2 u. \tag{2.8}$$

Condition (2.7), which is the same as the one in the last unnumbered equation on p. 257 in [8] can be established in the following way. Using (2.2), we have

$$u_r(r,\theta) = s_x(r\cos\theta, \sin\theta)\cos\theta + s_y(r\cos\theta, \sin\theta)\sin\theta,$$

and hence

$$\int_0^{2\pi} u_r(0,\theta)\mathrm{d}\theta = s_x(0,0)\int_0^{2\pi}\cos\theta\mathrm{d}\theta + s_y(0,0)\int_0^{2\pi}\sin\theta\mathrm{d}\theta = 0.$$

## 3. Spectral collocation scheme for the Dirichlet problem

Assume that $M$ and $N$ are positive integers. Let

$$V_\theta = \text{span}\{\psi_0(\theta), \ldots, \psi_{2M}(\theta)\},$$

where

$$\psi_0(\theta) = 1, \quad \psi_{2l-1}(\theta) = \cos(l\theta), \quad \psi_{2l}(\theta) = \sin(l\theta), \ l = 1, \ldots, M. \tag{3.1}$$

Let $P_N(0,R)$ be the set of all polynomials of degree $\leqslant N$ on $[0,R]$ and let

$$V = P_N(0,R) \otimes V_\theta, \quad \widetilde{V} = \{v \in V : v(0,\theta) = \text{const}, \ \theta \in [0,2\pi]\}.$$

Note that $\dim\widetilde{V} = N(2M+1) + 1$.

Let $\{\xi_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ be the nodes and weights of the $(N+1)$-point Chebyshev–Gauss–Lobatto quadrature for $[-1,1]$, respectively. Using (2.4.14) in [6], we have

$$\xi_i = \cos\frac{i\pi}{N}, \quad w_i = \frac{\pi}{N}, \quad i = 1, \ldots, N-1. \tag{3.2}$$

Let $\{r_i\}_{i=1}^{N-1}$ and $\{\theta_j\}_{j=0}^{2M}$ be the collocation points defined by

$$r_i = l^{-1}(\xi_i), \ \theta_j = \frac{2j\pi}{2M+1}, \tag{3.3}$$

where $l(r) = 2R^{-1}r - 1$ is the linear function mapping $[0,R]$ onto $[-1,1]$.

The spectral collocation problem corresponding to (2.4)–(2.7) consists of finding $U \in \widetilde{V}$ such that

$$LU(r_i,\theta_j) = f(r_i,\theta_j), \quad i = 1,\dots,N-1,\ j = 0,\dots,2M, \tag{3.4}$$

$$U(R,\theta_j) = g(\theta_j), \quad j = 0,\dots,2M, \tag{3.5}$$

$$\int_0^{2\pi} U_r(0,\theta)\mathrm{d}\theta = 0. \tag{3.6}$$

Note that the number of equations in (3.4)–(3.6) is equal to $\dim\widetilde{V}$.

## 4. Algorithm for solving the Dirichlet collocation problem

Following the idea of superposition (see (2.16)–(2.18) in [14]), we seek the solution $U$ of (3.4)–(3.6) in the form

$$U = U^{(1)} + cU^{(2)}, \tag{4.1}$$

where the constant $c$ is to be determined, and where $U^{(1)}$, $U^{(2)}$ in $V$ are such that

$$\begin{aligned} &LU^{(1)}(r_i,\theta_j) = f(r_i,\theta_j), \quad i = 1,\dots,N-1,\ j = 0,\dots,2M,\\ &U^{(1)}(0,\theta_j) = 0, \quad U^{(1)}(R,\theta_j) = g(\theta_j),\ j = 0,\dots,2M, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &LU^{(2)}(r_i,\theta_j) = 0, \quad i = 1,\dots,N-1,\ j = 0,\dots,2M,\\ &U^{(2)}(0,\theta_j) = 1, \quad U^{(2)}(R,\theta_j) = 0,\ j = 0,\dots,2M. \end{aligned} \tag{4.3}$$

Note that for any constant $c$, $U$ given by (4.1) is in $\widetilde{V}$, $U(0,\theta) = c$, $\theta \in [0,2\pi]$, and that $U$ satisfies (3.4) and (3.5). Clearly, in order for (3.6) to be satisfied, $c$ must be determined from

$$c = -\frac{\int_0^{2\pi} U_r^{(1)}(0,\theta)\mathrm{d}\theta}{\int_0^{2\pi} U_r^{(2)}(0,\theta)\mathrm{d}\theta}. \tag{4.4}$$

The solution $U^{(1)}$ in $V$ of (4.2) can be obtained as follows. Introduce

$$\widetilde{U}^{(1)}(r,\theta) = \frac{r}{R}\sum_{l=0}^{2M} \alpha_l \psi_l(\theta), \tag{4.5}$$

where $\{\alpha_l\}_{l=0}^{2M}$ are such that

$$\widetilde{U}^{(1)}(R,\theta_j) = \sum_{l=0}^{2M} \alpha_l \psi_l(\theta_j) = g(\theta_j), \quad j = 0,\dots,2M. \tag{4.6}$$

Let $\widehat{U}^{(1)} \in V$ be such that

$$\begin{aligned} &L\widehat{U}^{(1)}(r_i,\theta_j) = \hat{f}(r_i,\theta_j), \quad i = 1,\dots,N-1,\ j = 0,\dots,2M,\\ &\widehat{U}^{(1)}(0,\theta_j) = 0, \quad \widehat{U}^{(1)}(R,\theta_j) = 0,\ j = 0,\dots,2M, \end{aligned} \tag{4.7}$$

where $\hat{f}(r,\theta) = f(r,\theta) - L\widetilde{U}^{(1)}(r,\theta)$. Then

$$U^{(1)} = \widetilde{U}^{(1)} + \widehat{U}^{(1)} \tag{4.8}$$

is a solution of (4.2).

To solve (4.7), we introduce

$$\phi_k(r) = [1 - l^2(r)]T_{k-1}[l(r)], \quad k = 1,\dots,N-1, \tag{4.9}$$

where $T_{k-1}$ is the Chebyshev polynomial of degree $k-1$, that is (see, for example, (2.4.2) in [6]),

$$T_{k-1}(t) = \cos[(k-1)\cos^{-1} t], \quad t \in [-1,1]. \tag{4.10}$$

Then

$$\widehat{U}^{(1)}(r,\theta) = \sum_{k=1}^{N-1}\sum_{l=0}^{2M} u_{k,l}\phi_k(r)\psi_l(\theta), \tag{4.11}$$

where the coefficients $\{u_{k,l}\}_{k=1,l=0}^{N-1,2M}$ are to be determined from the first equation in (4.7).

We introduce

$$\mathbf{u} = [u_{1,0}, \ldots, u_{1,2M}, \ldots, u_{N-1,0}, \ldots, u_{N-1,2M}]^{\mathrm{T}},$$
$$\mathbf{f} = [f_{1,0}, \ldots, f_{1,2M}, \ldots, f_{N-1,0}, \ldots, f_{N-1,2M}]^{\mathrm{T}}, \; f_{i,j} = \hat{f}(r_i, \theta_j).$$

Substituting (4.11) into the first equation of (4.7) and using (2.8), we obtain

$$(A_r \otimes B_\theta + B_r \otimes A_\theta - \alpha C_r \otimes B_\theta)\mathbf{u} = \mathbf{f}, \tag{4.12}$$

where

$$A_r = (r_i^2 \phi_k''(r_i) + r_i \phi_k'(r_i))_{i,k=1}^{N-1}, \quad B_r = (\phi_k(r_i))_{i,k=1}^{N-1}, \quad C_r = (r_i^2 \phi_k(r_i))_{i,k=1}^{N-1}, \tag{4.13}$$
$$A_\theta = (\psi_l''(\theta_j))_{j,l=0}^{2M}, \quad B_\theta = (\psi_l(\theta_j))_{j,l=0}^{2M}, \tag{4.14}$$

and $i$, $j$ and $k$, $l$, are the row and column indices, respectively.

System (4.12) is solved using a matrix decomposition algorithm (see, for example, [3] and references within) and FFTs. Using (4.14), (3.1), and MATLAB notation, we have

$$A_\theta(:,0) = \mathbf{0}, \quad A_\theta(:,2l-1) = -l^2 B_\theta(:,2l-1), \quad A_\theta(:,2l) = -l^2 B_\theta(:,2l), \; l = 1, \ldots, M, \tag{4.15}$$

and hence

$$B_\theta^{-1} A_\theta = \Lambda = \operatorname{diag}(\lambda_0, \ldots, \lambda_{2M}), \tag{4.16}$$

where

$$\lambda_0 = 0, \quad \lambda_{2l-1} = \lambda_{2l} = -l^2, \; l = 1, \ldots, M.$$

It follows from (4.16) that (4.12) is equivalent to

$$(A_r \otimes I + B_r \otimes \Lambda - \alpha C_r \otimes I)\mathbf{u} = \mathbf{v}, \tag{4.17}$$

where $I$ is the identity matrix and $\mathbf{v} = (I \otimes B_\theta^{-1})\mathbf{f}$. Introducing

$$\mathbf{u}_j = [u_{1,j}, \ldots, u_{N-1,j}]^{\mathrm{T}}, \quad \mathbf{v}_j = [v_{1,j}, \ldots, v_{N-1,j}]^{\mathrm{T}}, \; j = 0, \ldots, 2M,$$

and using (4.16), we see that (4.17) becomes

$$(A_r + \lambda_j B_r - \alpha C_r)\mathbf{u}_j = \mathbf{v}_j, \quad j = 0, \ldots, 2M. \tag{4.18}$$

We introduce the following matrices:

$$W = \operatorname{diag}(w_1, \ldots, w_{N-1}), \quad D = \operatorname{diag}(1 + \xi_1, \ldots, 1 + \xi_{N-1}), \tag{4.19}$$
$$A = ((1 + \xi_i)\rho_k''(\xi_i) + \rho_k'(\xi_i))_{i,k=1}^{N-1}, \quad B = (\rho_k(\xi_i))_{i,k=1}^{N-1}, \tag{4.20}$$

where

$$\rho_k(t) = (1 - t^2)T_{k-1}(t), \quad k = 1, \ldots, N-1. \tag{4.21}$$

Since $\phi_k(r) = \rho_k(l(r))$, it is easy to show that

$$B^{\mathrm{T}} W D^{-1} A_r = A', \quad B^{\mathrm{T}} W D^{-1} B_r = B', \quad B^{\mathrm{T}} W D^{-1} C_r = C',$$

where

$$A' = B^{\mathrm{T}} W A, \quad B' = B^{\mathrm{T}} W D^{-1} B, \quad C' = \frac{R^2}{4} B^{\mathrm{T}} W D B. \tag{4.22}$$

Hence multiplying (4.18) by $B^{\mathrm{T}} W D^{-1}$, we obtain

$$(A' + \lambda_j B' - \alpha C')\mathbf{u}_j = B^{\mathrm{T}} W D^{-1} \mathbf{v}_j, \quad j = 0, \ldots, 2M. \tag{4.23}$$

It follows from Appendix A that the matrix $A' = (a'_{k,l})_{k,l=1}^{N-1}$ is heptadiagonal, $B' = (b'_{k,l})_{k,l=1}^{N-1}$ is symmetric heptadiagonal, $C' = (R^2/4)(c'_{k,l})_{k,l=1}^{N-1}$ is symmetric hendecadiagonal, and that their entries are given by following formulæ:

$$a'_{1,1} = -\pi, \quad a'_{1,2} = -5\pi/8, \quad a'_{1,4} = \pi/8,$$
$$a'_{2,1} = -\pi/2, \quad a'_{3,1} = a'_{4,1} = \pi/2, \quad a'_{3,2} = -\pi/4,$$
$$a'_{l,l} = -(\pi/4)(l^2 - 2l + 3), \quad l = 2, \ldots N-1,$$
$$a'_{l-1,l} = -(\pi/16)(l^2 - 2l + 5), \quad l = 3, \ldots N-1,$$
$$a'_{l-2,l} = (\pi/8)(l^2 - 5l + 6), \quad l = 3, \ldots N-1,$$

$$a'_{l-3,l} = (\pi/16)(l^2 - 6l + 9), \quad l = 5, \ldots N-1,$$
$$a'_{l+1,l} = -(\pi/16)(l^2 - 2l + 5), \quad l = 3, \ldots N-2,$$
$$a'_{l+2,l} = (\pi/8)(l^2 + l), \quad l = 2, \ldots N-3,$$
$$a'_{l+3,l} = (\pi/16)(l^2 + 2l + 1), \quad l = 2, \ldots N-4,$$
$$b'_{1,1} = \pi/2, \quad b'_{2,1} = -\pi/8, \quad b'_{3,1} = -\pi/4, \quad b'_{4,1} = \pi/8, \quad b'_{2,2} = \pi/8, \quad b'_{3,2} = 0,$$
$$b'_{l,l} = \pi/4,\ l = 3, \ldots N-1, \quad b'_{l,l-1} = -\pi/16,\ l = 4, \ldots N-1,$$
$$b'_{l,l-2} = -\pi/8,\ l = 4, \ldots N-1, \quad b'_{l,l-3} = \pi/16,\ l = 5, \ldots N-1,$$
$$c'_{1,1} = 3\pi/32, \quad c'_{2,1} = \pi/64, \quad c'_{3,1} = -\pi/16, \quad c'_{4,1} = -3\pi/128, \quad c'_{5,1} = \pi/64, \quad c'_{6,1} = \pi/128,$$
$$c'_{2,2} = \pi/64, \quad c'_{3,2} = -\pi/256, \quad c'_{4,2} = -3\pi/128, \quad c'_{5,2} = -\pi/128,$$
$$c'_{3,3} = 7\pi/128, \quad c'_{4,3} = 3\pi/256,$$
$$c'_{l,l} = 3\pi/64,\ l = 4, \ldots N-2, \quad c'_{l,l-1} = \pi/128,\ l = 5, \ldots N-1,$$
$$c'_{l,l-2} = -\pi/32,\ l = 5, \ldots N-1, \quad c'_{l,l-3} = -3\pi/256,\ l = 6, \ldots N-1,$$
$$c'_{l,l-4} = \pi/128,\ l = 6, \ldots N-1, \quad c'_{l,l-5} = \pi/256,\ l = 7, \ldots N-1.$$

The entries $c'_{N-1,N-1}$ and $c'_{N-1,N-2}$ are evaluated from their definitions (see formula (A.1) in Appendix A).

Let $z \in P_N(0,R)$ be such that

$$r_i^2 z''(r_i) + r_i z'(r_i) - \alpha r_i^2 z(r_i) = R^{-1} r_i + \alpha r_i^2 (1 - R^{-1} r_i), \quad i = 1, \ldots, N-1,\ z(0) = 0,\ z(R) = 0. \tag{4.24}$$

Then

$$U^{(2)}(r,\theta) = z(r) + 1 - R^{-1} r \tag{4.25}$$

is a solution of (4.3). With

$$z(r) = \sum_{k=1}^{N-1} \beta_k \phi_k(r), \tag{4.26}$$

the linear system corresponding to (4.24) is a special case of (4.18) with $j = 0$.

It follows from (4.4), (4.8), (4.5), (4.11), (3.1), (4.25), (4.26) and (4.9), and $T_{k-1}(-1) = (-1)^{k-1}$, that

$$c = \frac{R^{-1}\alpha_0 + \sum_{k=1}^{N-1} u_{k,0}\phi'_k(0)}{R^{-1} - \sum_{k=1}^{N-1}\beta_k\phi'_k(0)} = \frac{\alpha_0 - 4\sum_{k=1}^{N-1} u_{k,0}(-1)^k}{1 + 4\sum_{k=1}^{N-1}\beta_k(-1)^k}. \tag{4.27}$$

We arrive at the following algorithm for solving the collocation Dirichlet problem:

**Algorithm**

1. Solve the system in (4.6) to obtain $\widetilde{U}^{(1)}$ given by (4.5).
2. Solve the systems in (4.23) to obtain $\widehat{U}^{(1)}$ given by (4.11).
3. Solve (4.24) for $z$.
4. Compute the constant $c$ using (4.27).
5. Obtain $U$ using (4.1), (4.8), (4.25).

It follows from Appendix B that, with the help of FFTs, the cost of solving a system with the matrix $B_\theta$ of (4.14) is $O(M \log M)$. Also, with $B$ defined in (4.20), it follows from Appendix C that FFTs can be used to multiply a vector by $B^T$ at a cost $O(N \log N)$. Moreover, the cost of solving a hendecadiagonal linear system is proportional to the number of unknowns. Hence, for $N = M$, the cost of the algorithm is $O(N^2 \log N)$.

## 5. The Neumann collocation problem and its solution

We now consider boundary value problem (2.1) with a positive constant $\alpha$ and the Dirichlet boundary condition replaced with the Neumann boundary condition

$$\frac{\partial s}{\partial n} = q(x,y), \quad (x,y) \in \partial D, \tag{5.1}$$

where $\boldsymbol{n}$ is the outward unit normal vector to the boundary $\partial D$. Let $u$ and $f$, $g$ be defined as in (2.2) and (2.3), respectively. Since $\boldsymbol{n} = (x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}) = (\cos\theta, \sin\theta)$, we have

$$\frac{\partial s}{\partial n} = s_x \cos\theta + s_y \sin\theta = u_r \quad \text{on } \partial\Omega.$$

Hence we obtain problem (2.4)–(2.8) with the second equation in (2.5) replaced by $u_r(R,\theta) = g(\theta)$.

Let $\{\xi_i\}_{i=0}^{N-1}$ and $\{w_i\}_{i=0}^{N-1}$ be the nodes and weights of the $N$-point Chebyshev–Gauss–Radau quadrature for $[-1,1]$ with $\xi_0 = -1$. Using equation (1.3.10) in [12], we have

$$\xi_i = -\cos\frac{2\pi i}{2N-1}, \quad w_i = \frac{2\pi}{2N-1}, \; i = 1,\ldots,N-1. \tag{5.2}$$

With $\{\psi_l(\theta)\}_{l=0}^{2M}$ and $\{r_i\}_{i=1}^{N-1}$, $\{\theta_j\}_{j=0}^{2M}$ defined by equations in (3.1) and (3.3), respectively, the spectral collocation problem for the Neumann problem and the corresponding algorithm for its solution are similar to those for the Dirichlet problem. For example, in (3.5), (4.2) and (4.3), we now have

$$U_r(R,\theta_j) = g(\theta_j), \quad U_r^{(1)}(R,\theta_j) = g(\theta_j), \quad U_r^{(2)}(R,\theta_j) = 0, \; j = 0,\ldots,2M,$$

respectively. We require now that $\widetilde{U}^{(1)}$, defined in (4.5), satisfies

$$\widetilde{U}_r^{(1)}(R,\theta_j) = (1/R)\sum_{l=0}^{2M}\alpha_l\psi_l(\theta_j) = g(\theta_j), \quad j = 0,\ldots,2M,$$

and $\widehat{U}^{(1)}$ satisfies (4.7) but with $\widehat{U}^{(1)}(R,\theta_j) = 0$ replaced by $\widehat{U}_r^{(1)}(R,\theta_j) = 0$. Following [10], we introduce

$$\phi_k(r) = T_{k-1}(l(r)) + a_kT_k(l(r)) + b_kT_{k+1}(l(r)), \quad k = 1,\ldots,N-1, \tag{5.3}$$

where the $a_k$ and $b_k$ are selected so that $\phi_k(0) = \phi_k'(R) = 0$. Using $T_k(-1) = (-1)^k$, $T_k'(1) = k^2$, and $l'(r) = 2/R$, we obtain

$$1 - a_k + b_k = 0, \quad (k-1)^2 + a_kk^2 + b_k(k+1)^2 = 0,$$

whose solution is given by

$$a_k = \frac{4k}{2k^2+2k+1}, \quad b_k = \frac{-2k^2+2k-1}{2k^2+2k+1}.$$

With the $\phi_k$ defined in (5.3), finding the $u_{k,l}$ in (4.11) reduces to solving a system of the form (4.12) with the matrices $A_r$, $B_r$ and $C_r$ defined by the equations in (4.13). As before, the system is solved using the matrix decomposition algorithm and FFTs. With $W$, $D$ defined as in (4.19) and $A$, $B$ defined as in (4.20) but with

$$\rho_k(t) = T_{k-1}(t) + a_kT_k(t) + b_kT_{k+1}(t), \quad k = 1,\ldots,N-1, \tag{5.4}$$

we obtain (4.23), where $A'$, $B'$, and $C'$ are defined in (4.22). It follows from Appendix A that the matrix $A' = (a'_{k,l})_{k,l=1}^{N-1}$ is upper Hessenberg, $B' = (b'_{k,l})_{k,l=1}^{N-1}$ is symmetric tridiagonal, $C' = (R^2/4)(c'_{k,l})_{k,l=1}^{N-1}$ is symmetric heptadiagonal, and that their entries are given by following formulæ:

$$a'_{l+1,l} = -\frac{(l+1)^2(2l^2-2l+1)}{2l^2+2l+1}\pi, \quad l = 1,\ldots,N-2,$$
$$a'_{l,l} = -\frac{6l+2l^2-8l^3+8l^5+8l^6}{(2l^2+2l+1)^2}\pi, \quad l = 1,\ldots,N-2,$$
$$a'_{l,l+1} = -\frac{12l+37l^2+76l^3+64l^4+24l^5+4l^6}{(2l^2+2l+1)(2(l+1)^2+2(l+1)+1)}\pi, \quad l = 1,\ldots,N-2,$$
$$a'_{k,l} = -\frac{4k(3-4l^2+4l^4)}{(2k^2+2k+1)(2l^2+2l+1)}\pi, \quad k = 1,\ldots,N-3, \; l = k+2,\ldots,N-1,$$
$$b'_{l,l} = \left(1+\frac{(2l^2-2l+1)^2}{(2l^2+2l+1)^2}\right)\pi, \quad l = 1,\ldots,N-2,$$
$$b'_{l,l+1} = -\frac{2l^2-2l+1}{2l^2+2l+1}\pi, \quad l = 1,\ldots,N-2,$$
$$c'_{1,1} = \frac{103}{200}\pi, \quad c'_{l,l} = \frac{4l^4+24l^2+1}{4(2l^2+2l+1)^2}\pi, \; l = 2,\ldots,N-3,$$
$$c'_{1,2} = \frac{121}{520}\pi, \quad c'_{l,l+1} = \frac{4l^4+8l^3+88l^2+84l-7}{16(4l^4+16l^3+24l^2+16l+5)}\pi, \; l = 2,\ldots,N-3,$$
$$c'_{1,3} = \frac{19}{1000}\pi, \quad c'_{l,l+2} = -\frac{4l^4+16l^3-8l^2-48l+17}{4(8l^4+48l^3+96l^2+72l+26)}\pi, \; l = 2,\ldots,N-4,$$
$$c'_{l,l+3} = -\frac{2l^2-2l+1}{16(2l^2+2l+1)}\pi, \quad l = 1,\ldots,N-4.$$

The entries $a'_{N-1,N-1}$, $b'_{N-1,N-1}$, $c'_{N-2,N-2}$, $c'_{N-1,N-1}$, $c'_{N-2,N-1}$, and $c'_{N-3,N-1}$ are evaluated from their definitions.

We solve problem (4.3), with $U^{(2)}(R,\theta_j)=0$ replaced by $U_r^{(2)}(R,\theta_j)=0$, using an approach similar to that described in (4.24)–(4.26). Let $z\in P_N(0,R)$ be such that

$$r_i^2z''(r_i)+r_iz'(r_i)-\alpha r_i^2z(r_i)=\alpha r_i^2,\quad i=1,\ldots,N-1,\ z(0)=0,\ z'(R)=0. \tag{5.5}$$

Then $U^{(2)}(r,\theta)=z(r)+1$ is the required solution. With $z(r)=\sum_{k=1}^{N-1}\beta_k\phi_k(r)$, the linear system corresponding to (5.5) is a special case of (4.18) with $j=0$.

For the Neumann problem, the constant $c$ of (4.4) is given by

$$c=-\frac{R^{-1}\alpha_0+\sum_{k=1}^{N-1}u_{k,0}\phi_k'(0)}{\sum_{k=1}^{N-1}\beta_k\phi_k'(0)}=-\frac{\alpha_0+2\sum_{k=1}^{N-1}u_{k,0}\gamma_k}{2\sum_{k=1}^{N-1}\beta_k\gamma_k}, \tag{5.6}$$

where, by $T_k'(-1)=(-1)^{k-1}k^2$

$$\gamma_k=T_{k-1}'(-1)+a_kT_k'(-1)+b_kT_{k+1}'(-1)=(-1)^k[(k-1)^2-a_kk^2+b_k(k+1)^2].$$

Following [10] to solve (4.23) efficiently, we introduce

$$S=\mathrm{diag}(1,1,s_3,\ldots,s_{N-1}),\quad s_l=-\frac{2l^2+2l+1}{3-4l^2+4l^4},\ l=3,\ldots,N-1.$$

Solving any one of the systems in (4.23) is equivalent to solving

$$H\mathbf{w}=\mathbf{z}, \tag{5.7}$$

where

$$\mathbf{w}=S^{-1}\mathbf{u}_j,\quad \mathbf{z}=B^{\mathrm{T}}WD^{-1}\mathbf{v}_j,\quad H=(A'+\lambda_jB'-\alpha C')S.$$

As explained in Appendix C, a multiplication by $B^{\mathrm{T}}$ can be done using FFTs so that the cost of computing $\mathbf{z}$ is $O(N\log N)$. Since $A'$ is upper Hessenberg, $B'$ is tridiagonal, and $C'$ is heptadiagonal, the matrix $H=(h_{k,l})_{k,l=1}^{N-1}$ has zeros everywhere below its third subdiagonal. Also

$$h_{k,l}=\frac{4k\pi}{2k^2+2k+1},\quad k=1,\ldots,N-5,\ l=k+4,\ldots,N-1. \tag{5.8}$$

We solve (5.7) using Gauss elimination to create zeros on the first three subdiagonals of $H$. Taking advantage of (5.8), the cost of computing the resulting upper triangular matrix $U=(u_{k,l})_{k,l=1}^{N-1}$ is $O(N)$. Note that

$$u_{k,l}=u_k,\quad k=1,\ldots,N-5,\ l=k+4,\ldots,N-1, \tag{5.9}$$

where $u_k$ is a number depending only on $k$. Taking advantage of (5.9), the cost of applying back substitution is $O(N)$. All of this shows that once $\mathbf{z}$ has been computed, (5.7) can be solved at a cost $O(N)$. Hence, for $N=M$, the cost of solving the Neumann collocation problem is $O(N^2\log N)$.

## 6. Extension to variable coefficient problems

We consider the variable coefficient equation

$$c_1(x,y)s_{xx}+c_2(x,y)s_{yy}+c_{12}(x,y)s_{xy}+d_1(x,y)s_x+d_2(x,y)s_y-a(x,y)s=p(x,y),\quad (x,y)\in D, \tag{6.1}$$

with the Dirichlet boundary condition of (2.1). With $u$ and $f$, $g$ as in (2.2) and (2.3), and $\alpha(r,\theta)=a(r\cos\theta,r\sin\theta)$, we obtain (2.4)–(2.7), where now (see, for example, [5])

$$Lu=\gamma_1(r,\theta)r^2u_{rr}+\gamma_{12}(r,\theta)ru_{r\theta}+\gamma_2(r,\theta)u_{\theta\theta}+\delta_1(r,\theta)ru_r+\delta_2(r,\theta)ru_\theta-\alpha(r,\theta)r^2u,$$

and

$$\gamma_1(r,\theta)=c_1(r\cos\theta,r\sin\theta)\cos^2\theta+c_2(r\cos\theta,r\sin\theta)\sin^2\theta+(1/2)c_{12}(r\cos\theta,r\sin\theta)\sin(2\theta),$$

$$\gamma_{12}(r,\theta)=[c_2(r\cos\theta,r\sin\theta)-c_1(r\cos\theta,r\sin\theta)]\sin(2\theta)+c_{12}(r\cos\theta,r\sin\theta)\cos(2\theta),$$

$$\gamma_2(r,\theta)=c_1(r\cos\theta,r\sin\theta)\sin^2\theta+c_2(r\cos\theta,r\sin\theta)\cos^2\theta-(1/2)c_{12}(r\cos\theta,r\sin\theta)\sin(2\theta),$$

$$\delta_1(r,\theta)=c_1(r\cos\theta,r\sin\theta)\sin^2\theta+c_2(r\cos\theta,r\sin\theta)\cos^2\theta-(1/2)c_{12}(r\cos\theta,r\sin\theta)\sin(2\theta)+d_1(r\cos\theta,r\sin\theta)r\cos\theta$$
$$+d_2(r\cos\theta,r\sin\theta)r\sin\theta,$$

$$\delta_2(r,\theta) = [c_1(r\cos\theta, r\sin\theta) - c_2(r\cos\theta, r\sin\theta)]\sin(2\theta) - c_{12}(r\cos\theta, r\sin\theta)\cos(2\theta) - d_1(r\cos\theta, r\sin\theta)\sin\theta + d_2(r\cos\theta, r\sin\theta)\cos\theta.$$

Following the steps in (4.1)–(4.11), we see that the $u_{k,l}$ in (4.11) are determined from

$$[D(\gamma_1)A_r^{(1)} \otimes B_\theta + D(\gamma_{12})A_r^{(2)} \otimes C_\theta + D(\gamma_2)B_r \otimes A_\theta + D(\delta_1)A_r^{(2)} \otimes B_\theta + D(\delta_2)A_r^{(3)} \otimes C_\theta - D(\alpha)C_r \otimes B_\theta]\mathbf{u} = \mathbf{f}, \tag{6.2}$$

where $B_r$ and $C_r$ are defined in (4.13), $A_\theta$ and $B_\theta$ are defined in (4.14),

$$A_r^{(1)} = (r_i^2\phi_k''(r_i))_{i,k=1}^{N-1}, \quad A_r^{(2)} = (r_i\phi_k'(r_i))_{i,k=1}^{N-1}, \quad A_r^{(3)} = (r_i\phi_k(r_i))_{i,k=1}^{N-1}, \quad C_\theta = (\psi_l'(\theta_j))_{j,l=0}^{2M}, \tag{6.3}$$

and

$$D(\gamma) = \mathrm{diag}(\gamma(r_1,\theta_0),\dots,\gamma(r_1,\theta_{2M}),\dots,\gamma(r_{N-1},\theta_0),\dots,\gamma(r_{N-1},\theta_{2M})).$$

Using (6.3), (3.1), (4.14), and MATLAB notation, we have

$$C_\theta(:,0) = \mathbf{0}, \quad C_\theta(:,2l-1) = -lB_\theta(:,2l), \quad C_\theta(:,2l) = lB_\theta(:,2l-1), \; l = 1,\dots,M. \tag{6.4}$$

To compute $U^{(2)}$ satisfying (4.3), we use (4.24) with $z(r)$ replaced by $z(r,\theta)$. The $z_{k,l}$ in

$$z(r,\theta) = \sum_{k=1}^{N-1}\sum_{l=0}^{2M} z_{k,l}\phi_k(r)\psi_l(\theta). \tag{6.5}$$

are determined by solving the system (6.2), where the entries of the right-hand side are given by $f_{i,j} = R^{-1}\delta_1(r_i,\theta_j)r_i + \alpha(r_i,\theta_j)r_i^2(1 - R^{-1}r_i)$. The constant $c$ of (4.4) is given by (4.27) with $\beta_k$ replaced by $z_{k,0}$.

For the Neumann boundary condition (5.1), the determination of $U^{(1)}$ is similar (see also Section 5). Of course, in this case, we use the $\xi_i$ and $w_i$ given in (5.2) and the $\phi_k(r)$ defined by (5.3). To compute $U^{(2)}$, we use (5.5) with $z(r)$ replaced by $z(r,\theta)$. The $z_{k,l}$ in (6.5) are determined by solving the system (6.2), where the entries of the right-hand side are given by $f_{i,j} = \alpha(r_i,\theta_j)r_i^2$. The constant $c$ of (4.4) is given by (5.6) with $\beta_k$ replaced by $z_{k,0}$.

For the Dirichlet and Neumann problems, the system (6.2) can be solved using the preconditioned GMRES (see, for example, [15]) with the preconditioner equal to the matrix corresponding to the Helmholtz equation with $\alpha = 0$ and $\alpha = 1$ for the Dirichlet and Neumann problems, respectively. A linear system with the preconditioner can be solved very efficiently using the FFT matrix decomposition algorithms of Sections 4 and 5. As explained in Appendices B and D, FFTs can also be used to multiply a vector by the matrix in (6.2). This results in a very efficient overall procedure whose cost depends mainly on the number of the iterations in the preconditioned GMRES.

## 7. Numerical results

In the first seven examples, we tested accuracy of the spectral collocation method for Dirichlet and Neumann problems. Following [8], for various values of $N$, we chose $M = N$ and calculated the maximum error $E_c$ at the collocation points and the maximum global error $E_g$ on a $100 \times 201$ uniform grid on $[0,R] \times [0,2\pi]$. In the first four examples, our errors $E_c$ are close

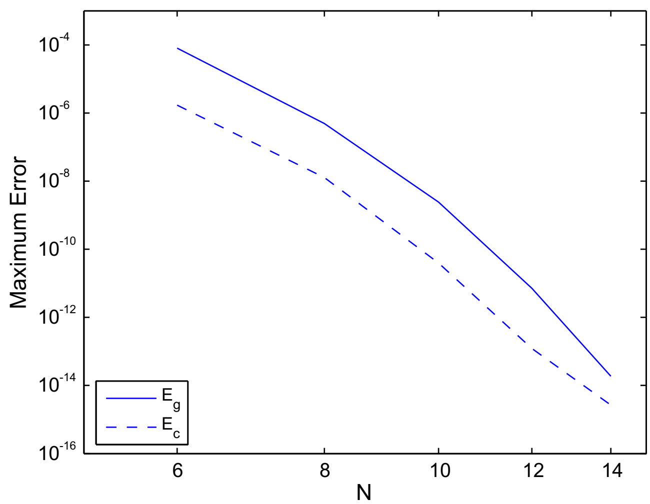


**Fig. 1.** Plots for the errors in Example 1.

to those reported in [8]. The last two examples are concerned with the condition number and CPU times. In all examples $D$ was taken to be the unit disk.

**Example 1.** We considered the Poisson equation ($\alpha = 0$) with exact solution $u(r,\theta) = e^{r(\cos\theta+\sin\theta)}$ subject to a Dirichlet boundary condition. This example was also considered in [8]. In Fig. 1, we present the log–log plots of $E_c$ and $E_g$.

**Example 2.** We considered the Helmholtz equation with $\alpha = 30$ with exact solution $u(r,\theta) = \cos(3r\cos\theta + 4r\sin\theta + 0.7)$ subject to a Dirichlet boundary condition. This example was also considered in [8]. In Fig. 2, we present the log–log plots of $E_c$ and $E_g$.

**Example 3.** We considered the Dirichlet problem for the Poisson equation with exact solution $u(r,\theta) = \cos(7r\cos\theta + 8r\sin\theta + 0.7)$. This example was also considered in [8]. In Fig. 3, we present the log–log plots of $E_c$ and $E_g$.

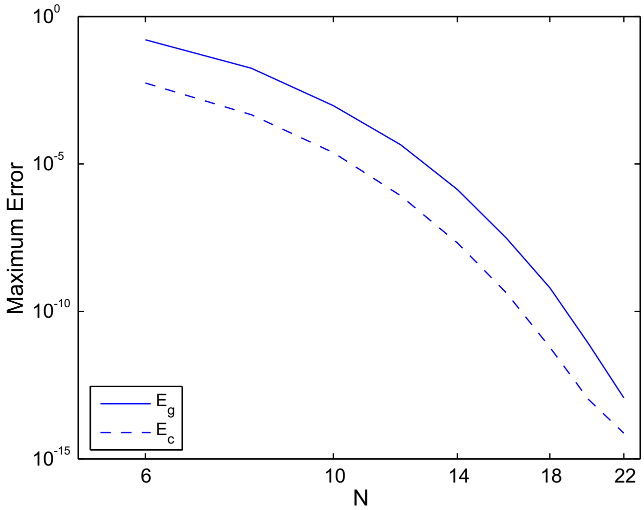


**Fig. 2.** Plots for the errors in Example 2.

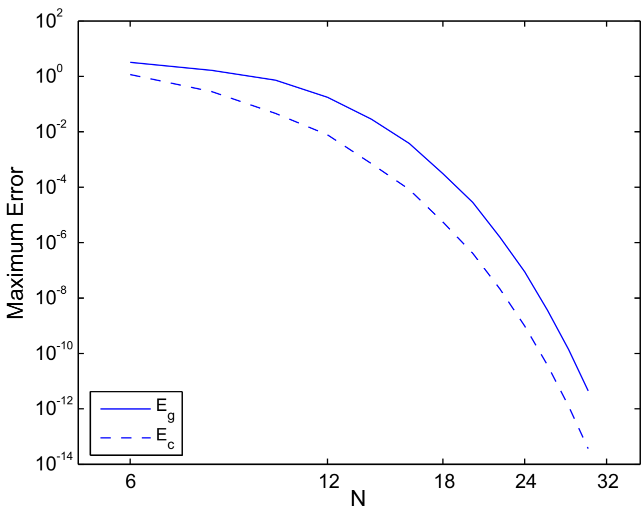


**Fig. 3.** Plots for the errors in Example 3.

**Example 4.** We considered the Poisson equation with exact solution $u(r,\theta) = r^{3.5}$ subject to a Dirichlet boundary condition. This example was also considered in [8] to demonstrate that the spectral collocation method is more accurate than the spectral tau method of [7]. In Fig. 4, we present the log–log plots of $E_c$ and $E_g$.

**Example 5.** We considered the Helmholtz equation with $\alpha = 30$ with exact solution $u(r,\theta) = \cos(3r\cos\theta + 4r\sin\theta + 0.7)$ subject to a Neumann boundary condition. In Fig. 5, we present the log–log plots of $E_c$ and $E_g$.

**Example 6.** We considered Eq. (6.1) with the variable coefficients

$$c_1(x,y) = 1, \quad c_2(x,y) = e^{xy}, \quad c_{12}(x,y) = (x+y)/10, \quad d_1(x,y) = x^2 + y^2, \quad d_2(x,y) = \cos(x+y), \quad a(x,y) = 1 + \sin(xy),$$

and the exact solution $u(r,\theta) = \cos(3r\cos\theta + 4r\sin\theta + 0.7)$ subject to a Dirichlet boundary condition. In Fig. 6, we present the log–log plots of $E_c$ and $E_g$. These results were obtained using the preconditioned GMRES with number of iterations equal to $5\log_2 N$.

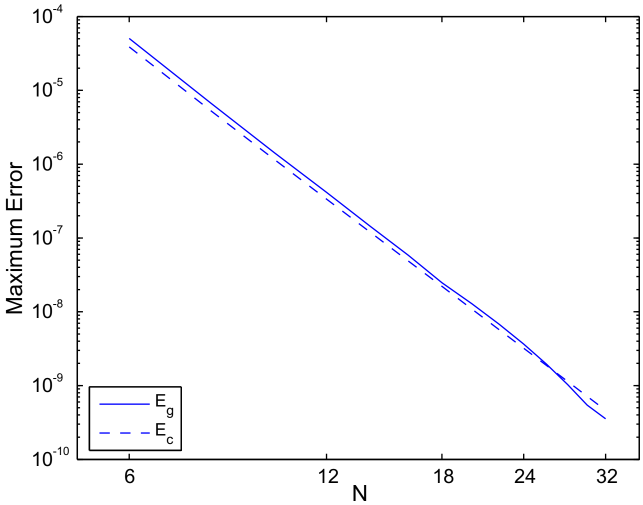


**Fig. 4.** Plots for the errors in Example 4.

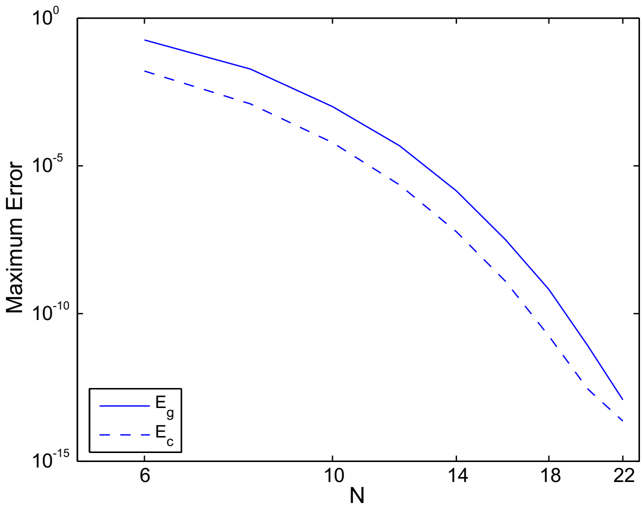


**Fig. 5.** Plots for the errors in Example 5.

**Example 7.** We considered Eq. (6.1) with the same variable coefficients and the same exact solution as in Example 6, but subject to a Neumann boundary condition. In Fig. 7, we present the log–log plots of $E_c$ and $E_g$. These results were obtained using the preconditioned GMRES with number of iterations equal to $5\log_2 N$.

**Example 8.** In Fig. 8, we present the two-norm condition number $\kappa_2$ of the matrix in (4.12) corresponding to the Dirichlet problem for the Helmholtz equation with $\alpha = 0$ and $\alpha = 30$. From this figure, it appears that the condition number behaves like $O(N^4)$ which is consistent with similar observations in the literature. For Cartesian geometries, in [12, pp. 87–88], it is mentioned that the Chebyshev collocation matrix involved in the spectral solution of second order problems (for collocation at the Gauss–Lobatto points) has a condition number proportional to $N^4$. In [9, pp. 60–62], the same matrices have condition numbers $O(N^4)$ and $O(N^{9/2})$; the order is $N^{9/2}$ when the boundary values are not eliminated from the system. In [4, pp. 142–145], it is stated that the condition numbers of the Chebyshev or Legendre collocation matrices are of order $2p$, where $p$ is the order of the derivative involved. For axisymmetric problems, in [2, pp. 153–154], the matrices corresponding to the radial part of the Laplace operator are found to behave like $O(N^4)$ for Legendre collocation and like $O(N^3)$ for the Legendre Galerkin method. This behavior for the Legendre Galerkin method is consistent with the theoretical result for the corresponding matrices in the Cartesian case [1, p. 104]. Despite the fact that the condition number grows like $O(N^4)$, we obtained very good accuracy in all our examples which suggests that the condition number has a modest effect on round-off errors.

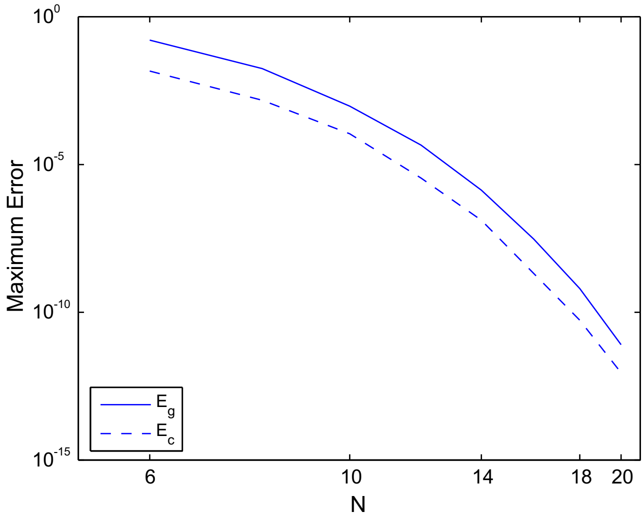


**Fig. 6.** Plots for the errors in Example 6.

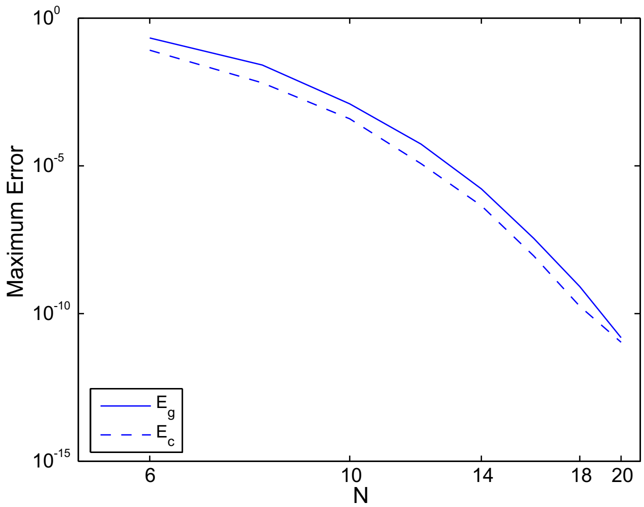


**Fig. 7.** Plots for the errors in Example 7.

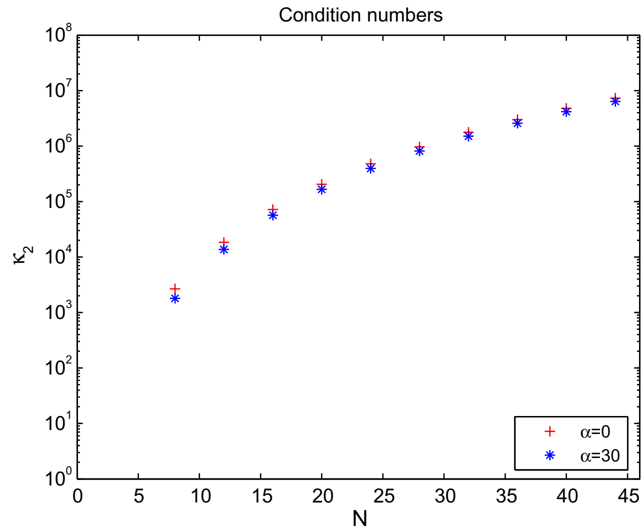


**Fig. 8.** Condition number $\kappa_2$ of matrix in (4.12) for $\alpha = 0$ and $\alpha = 30$.

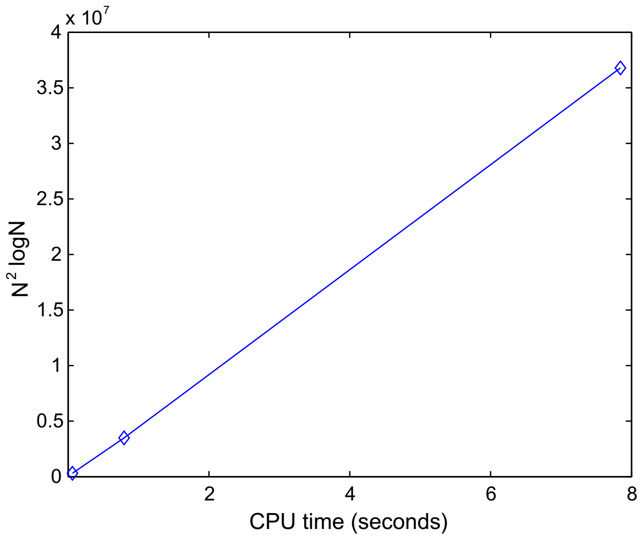


**Fig. 9.** CPU times versus $N^2 \log N$.

**Example 9.** In Fig. 9, we present the plot of the CPU times for the algorithm of Section 4. These times were recorded on an IBM PC (Processor: Intel Pentium 4, 3.4 GHz) for values of $N$ and $M$ for which the FFT routines are most efficient. We took $N = 2M + 1$ for $M = 121,364,1093$, so that $N$ was equal to powers of 3. We present the plot of the CPU times versus the corresponding values of $N^2 \log N$, from which we observe the linear dependence of the two quantities.

## 8. Concluding remarks

In this study, we present an efficient spectral collocation method for the solution of the Dirichlet and Neumann problems for the Helmholtz equation in a disk. The formulation of the collocation problem appears to be simpler than the corresponding formulations of the Galerkin and tau schemes. With the number of unknowns equal to $O(N^2)$, the cost of the proposed FFT matrix decomposition algorithm for solving the collocation problem is $O(N^2 \log N)$. Extensions to equations with variable coefficients are also given. The numerical results demonstrate the spectral accuracy of the method.

## Appendix A

With $A' = (a'_{k,l})_{k,l=1}^{N-1}$, $B' = (b'_{k,l})_{k,l=1}^{N-1}$, and $C' = (R^2/4)(c'_{k,l})_{k,l=1}^{N-1}$ defined as in (4.22), we show how to derive the formulæ for $a'_{k,l}$, $b'_{k,l}$, $c'_{k,l}$ in the case of the Dirichlet problem; the derivations for the Neumann problem are similar.

It follows from (4.22), (4.19), and (4.20) that

$$a'_{k,l} = \sum_{i=1}^{N-1} w_i \rho_k(\xi_i)[(1+\xi_i)\rho''_l(\xi_i) + \rho'_l(\xi_i)], \quad b'_{k,l} = \sum_{i=1}^{N-1} w_i (1+\xi_i)^{-1} \rho_k(\xi_i)\rho_l(\xi_i),$$
$$c'_{k,l} = \sum_{i=1}^{N-1} w_i (1+\xi_i)\rho_k(\xi_i)\rho_l(\xi_i). \tag{A.1}$$

With $\omega(t) = (1-t^2)^{-1/2}$, the exactness of the $N+1$-point Chebyshev–Gauss–Lobatto quadrature for polynomials of degree $\leqslant 2N-1$ gives

$$a'_{k,l} = \int_{-1}^{1} \rho_k(t)[(1+t)\rho''_l(t) + \rho'_l(t)]\omega(t)\mathrm{d}t, \quad b'_{k,l} = \int_{-1}^{1} (1+t)^{-1}\rho_k(t)\rho_l(t)\omega(t)\mathrm{d}t \tag{A.2}$$

for $k, l = 1,\ldots,N-1$, and

$$c'_{k,l} = \int_{-1}^{1} (1+t)\rho_k(t)\rho_l(t)\omega(t)\mathrm{d}t \tag{A.3}$$

for $k + l \leqslant 2N-4$. The integrals in (A.2) and (A.3) also arise in the spectral Chebyshev Galerkin method of [10], where it is also stated that the resulting matrices $A'$ and $B'$ are heptadiagonal and the matrix $C'$ is hendecadiagonal. Since the explicit formulæ for the integrals in (A.2) and (A.3) are not given in [10], we derive these formulæ in the following.

Using (4.21), we get

$$\rho'_l(t) = (1-t^2)T'_{l-1}(t) - 2tT_{l-1}(t), \quad \rho''_l(t) = (1-t^2)T''_{l-1}(t) - 4tT'_{l-1}(t) - 2T_{l-1}(t).$$

Substitution into $a'_{k,l}$ and simplifications yield

$$a'_{k,l} = I_1 + I_2 + I_3,$$

where

$$I_1 = \int_{-1}^{1} (1-t^2)(1+t)\rho_k(t)T''_{l-1}(t)\omega(t)\mathrm{d}t,$$
$$I_2 = \int_{-1}^{1} (1-4t-5t^2)\rho_k(t)T'_{l-1}(t)\omega(t)\mathrm{d}t,$$
$$I_3 = -2\int_{-1}^{1} (1+2t)\rho_k(t)T_{l-1}(t)\omega(t)\mathrm{d}t.$$

Using (4.10), we have

$$T'_{l-1}(t) = (l-1)(1-t^2)^{-1/2}\sin((l-1)\cos^{-1}t), \tag{A.4}$$
$$T''_{l-1}(t) = (l-1)t(1-t^2)^{-3/2}\sin((l-1)\cos^{-1}t) - (l-1)^2(1-t^2)^{-1}\cos((l-1)\cos^{-1}t). \tag{A.5}$$

Substitution into $I_2$ and $I_1$, and use of (4.21) along with simplifications give

$$I_1 = (l-1)\int_{-1}^{1} t(1-t^2)^{1/2}(1+t)T_{k-1}(t)\sin((l-1)\cos^{-1}t)\omega(t)\mathrm{d}t$$
$$- (l-1)^2\int_{-1}^{1} (1-t^2)(1+t)T_{k-1}(t)\cos((l-1)\cos^{-1}t)\omega(t)\mathrm{d}t,$$

$$I_2 = (l-1)\int_{-1}^{1} (1-t^2)^{1/2}(1-4t-5t^2)T_{k-1}(t)\sin((l-1)\cos^{-1}t)\omega(t)\mathrm{d}t,$$

$$I_3 = -2\int_{-1}^{1} (1-t^2)(1+2t)T_{k-1}(t)T_{l-1}(t)\omega(t)\mathrm{d}t.$$

Making the change of variable $t = \cos x$, we get

$$I_1 = (l-1)\int_0^{\pi} \sin x(\cos x + \cos^2 x)\cos((k-1)x)\sin((l-1)x)\mathrm{d}x$$
$$- (l-1)^2\int_0^{\pi} \sin^2 x(1+\cos x)\cos((k-1)x)\cos((l-1)x)\mathrm{d}x,$$

$$I_2 = (l-1)\int_0^{\pi} \sin x(1-4\cos x-5\cos^2 x)\cos((k-1)x)\sin((l-1)x)\mathrm{d}x,$$

$$I_3 = -2\int_0^{\pi} \sin^2 x(1+2\cos x)\cos((k-1)x)\cos((l-1)x)\mathrm{d}x.$$

In a similar way, we obtain

$$b'_{k,l} = \int_0^{\pi} \sin^2 x(1-\cos x)\cos((k-1)x)\cos((l-1)x)\mathrm{d}x,$$

$$c'_{k,l} = \int_0^{\pi} \sin^4 x(1+\cos x)\cos((k-1)x)\cos((l-1)x)\mathrm{d}x.$$

We obtained the formulas for $a'_{k,l}$, $b'_{k,l}$, and $c'_{k,l}$ given in Section 3 by computing all required integrals, which are of the form

$$I(n,\alpha,\beta,k,l) = \int_0^{\pi} \sin^n x(1+\alpha\cos x+\beta\cos^2 x)\cos((k-1)x)\cos((l-1)x)\mathrm{d}x,$$

using the Symbolic Math Toolbox of MATLAB. For example, to evaluate the integral $I(4, 1, 0, 1, 2)$, we used

```
≫ syms x y l k
≫ y=sin(x)^4*(1 + cos(x));
≫ k = 1; l = 2;
≫ int(y*cos((l-1)*x)*cos((k-1)*x),0,pi)
ans=
1/16*pi
```

## Appendix B

Let $A_\theta$, $B_\theta$, and $C_\theta$ be the matrices defined in (4.14) and (6.3). Then multiplying a vector by $B_\theta$ is equivalent to the following: given numbers $(a_l)_{l=0}^M$ and $(b_l)_{l=1}^M$, find numbers $(d_j)_{j=0}^{2M}$ such that

$$a_0 + \sum_{l=1}^{M} a_l \cos\frac{2lj\pi}{2M+1} + \sum_{l=1}^{M} b_l \sin\frac{2lj\pi}{2M+1} = d_j, \quad j = 0,\ldots,2M. \tag{B.1}$$

The numbers $d_j$ can be computed at a cost $O(M\log M)$ using the subroutine rfftb of [13].

Solving a linear system with the matrix $B_\theta$ is equivalent to the following: given numbers $(d_j)_{j=0}^{2M}$, find numbers $(a_l)_{l=0}^M$ and $(b_l)_{l=1}^M$ such that (B.1) holds. The numbers $a_l$ and $b_l$ can be computed at a cost $O(M\log M)$ using the subroutine rfftf of [13] since this subroutine is an unnormalized inverse of rfftb.

It follows from (4.15) and (6.4) that multiplying a vector by $A_\theta$ or $C_\theta$ is equivalent to multiplying a related vector by $B_\theta$. Hence multiplications by $A_\theta$ and $B_\theta$ can be carried out, each at a cost $O(M\log M)$, using the subroutine rfftb of [13].

## Appendix C

In the case of the Dirichlet problem, it follows from (4.20) and (4.21) that multiplication by $B^{\mathrm{T}}$ involves multiplication by the transpose of $(T_{k-1}(\xi_i))_{i,k=1}^{N-1}$. For $\mathbf{v} = [v_1,\ldots,v_{N-1}]^{\mathrm{T}}$ and $k = 1,\ldots,N-1$, using (3.2) and (4.10), we have

$$\sum_{i=1}^{N-1} T_{k-1}(\xi_i)v_i = \sum_{i=1}^{N-1} v_i \cos\frac{(k-1)i\pi}{N}. \tag{C.1}$$

Since

$$\cos\frac{(k-1)i\pi}{N} = \cos\left[\frac{(2k-1)i\pi}{2N} - \frac{i\pi}{2N}\right] = \cos\frac{(2k-1)i\pi}{2N}\cos\frac{i\pi}{2N} + \sin\frac{(2k-1)i\pi}{2N}\sin\frac{i\pi}{2N},$$

we have

$$\sum_{i=1}^{N-1} T_{k-1}(\xi_i)v_i = \sum_{i=1}^{N-1} \tilde{v}_i \cos\frac{(2k-1)i\pi}{2N} + \sum_{i=1}^{N-1} \hat{v}_i \sin\frac{(2k-1)i\pi}{2N},$$

where $\tilde{v}_i = v_i\cos\frac{i\pi}{2N}$ and $\hat{v}_i = v_i\sin\frac{i\pi}{2N}$. For $k = 1,\ldots,N-1$, the required sums can be computed, at a cost $O(N\log N)$, using the subroutines cosqf and sinqf of [13].

In the case of the Neumann problem, it follows from (4.20) and (5.4) that multiplication by $B^{\mathrm{T}}$ involves multiplication by the transposes of $(T_{k-1}(\xi_i))_{i,k=1}^{N-1}$, $(T_k(\xi_i))_{i,k=1}^{N-1}$, and $(T_{k+1}(\xi_i))_{i,k=1}^{N-1}$. Using (5.2), $-\cos t = \cos(\pi + t)$, and (4.10), we have

$$\sum_{i=1}^{N-1} T_{k-1}(\xi_i)v_i = \sum_{i=1}^{N-1} v_i \cos\left[(k-1)\pi + \frac{2(k-1)i\pi}{2N-1}\right] = (-1)^{k-1}\sum_{i=1}^{N-1} v_i \cos\frac{2(k-1)i\pi}{2N-1}, \tag{C.2}$$

which, for $k = 1,\ldots,N-1$, can be computed, at a cost $O(N\log N)$, using the subroutine `rfftf` or `rfftb` of [13]. Similarly, for $k = 1,\ldots,N-1$, we can compute

$$\sum_{i=1}^{N-1} T_k(\xi_i)v_i = (-1)^k \sum_{i=1}^{N-1} v_i \cos\frac{2ki\pi}{2N-1}, \quad \sum_{i=1}^{N-1} T_{k+1}(\xi_i)v_i = (-1)^{k+1}\sum_{i=1}^{N-1} v_i \cos\frac{2(k+1)i\pi}{2N-1},$$

at a cost $O(N^2\log N)$, using the subroutine `rfftf` or `rfftb` of [13].

## Appendix D

For the Dirichlet problem, it follows from (4.9) that multiplications by $B_r$, $C_r$ of (4.13) and $A_r^{(1)}$, $A_r^{(2)}$, $A_r^{(3)}$ of (6.3) involve multiplications by the matrices $(T_{k-1}(\xi_i))_{i,k=1}^{N-1}$, $(T'_{k-1}(\xi_i))_{i,k=1}^{N-1}$, and $(T''_{k-1}(\xi_i))_{i,k=1}^{N-1}$. Multiplication of a vector $\mathbf{v} = [v_1,\ldots,v_{N-1}]^T$ by $(T_{k-1}(\xi_i))_{i,k=1}^{N-1}$ yields, for $i = 1,\ldots,N-1$, (compare (C.1))

$$\sum_{k=1}^{N-1} T_{k-1}(\xi_i)v_k = \sum_{k=1}^{N-1} v_k \cos\frac{(k-1)i\pi}{N}.$$

Since

$$\cos\frac{(k-1)i\pi}{N} = \cos\left[\frac{(k-1)(2i-1)\pi}{2N} + \frac{(k-1)\pi}{2N}\right]$$
$$= \cos\frac{(k-1)(2i-1)\pi}{2N}\cos\frac{(k-1)\pi}{2N} - \sin\frac{(k-1)(2i-1)\pi}{2N}\sin\frac{(k-1)\pi}{2N},$$

we have

$$\sum_{k=1}^{N-1} T_{k-1}(\xi_i)v_k = \sum_{k=1}^{N-1} \tilde{v}_k \cos\frac{(k-1)(2i-1)\pi}{2N} - \sum_{k=1}^{N-1} \hat{v}_k \sin\frac{(k-1)(2i-1)\pi}{2N},$$

where $\tilde{v}_k = v_k\cos\frac{(k-1)\pi}{2N}$ and $\hat{v}_k = v_k \sin\frac{(k-1)\pi}{2N}$. For $i = 1,\ldots,N-1$, the required sums can be computed, at a cost $O(N\log N)$, using the subroutines `cosqf` and `sinqf` of [13]. By (A.4), multiplication of $\mathbf{v}$ by $(T'_{k-1}(\xi_i))_{i,k=1}^{N-1}$ yields, for $i = 1,\ldots,N-1$,

$$\sum_{k=1}^{N-1} T'_{k-1}(\xi_i)v_k = (1-\xi_i^2)^{-1/2}\sum_{k=1}^{N-1}(k-1)v_k \sin\frac{(k-1)i\pi}{N}.$$

Since

$$\sin\frac{(k-1)i\pi}{N} = \sin\frac{(k-1)(2i-1)\pi}{2N}\cos\frac{(k-1)\pi}{2N} + \cos\frac{(k-1)(2i-1)\pi}{2N}\sin\frac{(k-1)\pi}{2N},$$

it can be seen, as in the case above, that the required sums may be computed, at a cost $O(N\log N)$, using the subroutines `cosqf` and `sinqf` of [13]. By (A.5), multiplication of $\mathbf{v}$ by $(T''_{k-1}(\xi_i))_{i,k=1}^{N-1}$ involves sums of the forms $\sum_{k=1}^{N-1} w_k \sin\frac{(k-1)i\pi}{N}$ and $\sum_{k=1}^{N-1} z_k \cos\frac{(k-1)i\pi}{N}$ with appropriately defined $w_k$ and $z_k$. It can be seen, as in the two cases above, that the required sums may be computed, at a cost $O(N\log N)$, using the subroutines `cosqf` and `sinqf` of [13].

For the Neumann problem, it follows from (5.3) that multiplications by $B_r$, $C_r$ of (4.13) and $A_r^{(1)}$, $A_r^{(2)}$, $A_r^{(3)}$ of (6.3) involve multiplications by the matrices $(T_{k-1}(\xi_i))_{i,k=1}^{N-1}$, $(T'_{k-1}(\xi_i))_{i,k=1}^{N-1}$, $(T''_{k-1}(\xi_i))_{i,k=1}^{N-1}$ and the same matrices with the subscript $k-1$ on $T$ replaced by $k$ and $k+1$. Multiplication of $\mathbf{v}$ by $(T_{k-1}(\xi_i))_{i,k=1}^{N-1}$ yields, for $i = 1,\ldots,N-1$, (compare (C.2))

$$\sum_{k=1}^{N-1} T_{k-1}(\xi_i)v_k = \sum_{k=1}^{N-1}(-1)^{k-1}v_k \cos\frac{2(k-1)i\pi}{2N-1},$$

which, for $i = 1,\ldots,N-1$, can be calculated, at a cost $O(N\log N)$, using the subroutine `rfftf` or `rfftb` of [13]. By (A.4), multiplication of $\mathbf{v}$ by $(T'_{k-1}(\xi_i))_{i,k=1}^{N-1}$ yields, for $i = 1,\ldots,N-1$

$$\sum_{k=1}^{N-1} T'_{k-1}(\xi_i)v_k = (1-\xi_i^2)^{-1/2}\sum_{k=1}^{N-1}(k-1)v_k \sin\left((k-1)\pi + \frac{2(k-1)i\pi}{2N-1}\right) = (1-\xi_i^2)^{-1/2}\sum_{k=1}^{N-1}\tilde{v}_k \sin\frac{2(k-1)i\pi}{2N-1},$$

where $\tilde{v}_k = v_k(k-1)(-1)^{k-1}$. Hence the required sums can be calculated, at a cost $O(N^2\log N)$, using the subroutine `rfftf` or `rfftb` of [13]. By (A.5), multiplication of $\mathbf{v}$ by $(T''_{k-1}(\xi_i))_{i,k=1}^{N-1}$ yields sums of the forms $\sum_{k=1}^{N-1} w_k \sin\frac{2(k-1)i\pi}{2N-1}$ and $\sum_{k=1}^{N-1} z_k \cos\frac{2(k-1)i\pi}{2N-1}$ with appropriately defined $w_k$ and $z_k$. The required sums can be evaluated, at a cost $O(N^2\log N)$, using the subroutine `rfftb` or `rfftf` of [13].

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